CSCI 3210:
Computational Game Theory

Linear Programming and Game Theory

1. Intro to Linear Programming and Game Theory book on Canvas: Ch 1, 2, and 4
2. [AGT] Ch 1

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2-player zero-sum game

Prove that NE exists– in two ways

1. Nash's theorem
   - Doesn't give an algorithm (why?)
2. Linear programming
   - Gives an algorithm

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Linear Programming (LP)

Will come back to game theory later

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Applications

- Production, machine scheduling, employee scheduling, supply chain management, etc.
- Game theory
- In general: optimization

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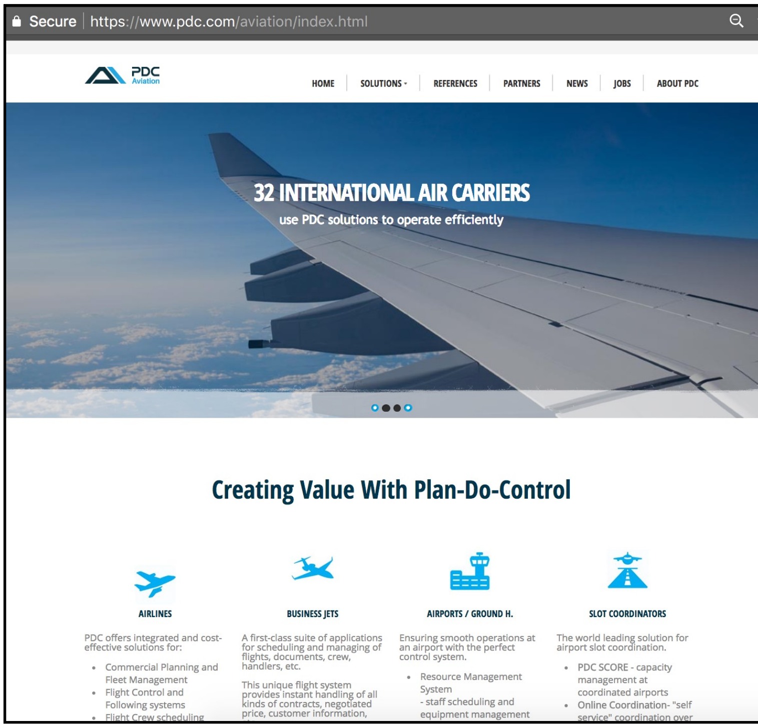

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LP

1. Variables (or decision variables)
   - We can choose the values of these variables
   - *What's the use of it?*
   - *What range of values can we choose? Integer vs real? Any other restrictions?*
2. Objective function (What's the goal?)
   - Minimization or maximization
   - Must be linear in the variables
3. Constraints (What values?)
   - Restricts the values of choice variables
   - Must be linear in the variables

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Example 1: diet problem

- A nutritionist wants to prepare a special diet for a patient. The meals should contain a minimum of 400 mg of calcium, 10 mg of iron, and 40 mg of vitamin C. The meals are to be prepared from foods A and B.
    - Each ounce of food A contains 30 mg of calcium, 1 mg of iron, 2 mg of vitamin C, and 2 mg of cholesterol.
    - Each ounce of food B contains 25 mg of calcium, 0.5 mg of iron, 5 mg of vitamin C, and 4 mg of cholesterol.
- How many ounces of A and B should be used so that the cholesterol content is minimized and the minimum requirements of calcium, iron, and vitamin C are met?

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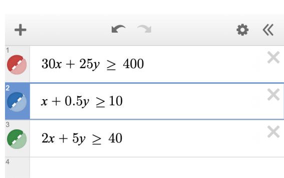
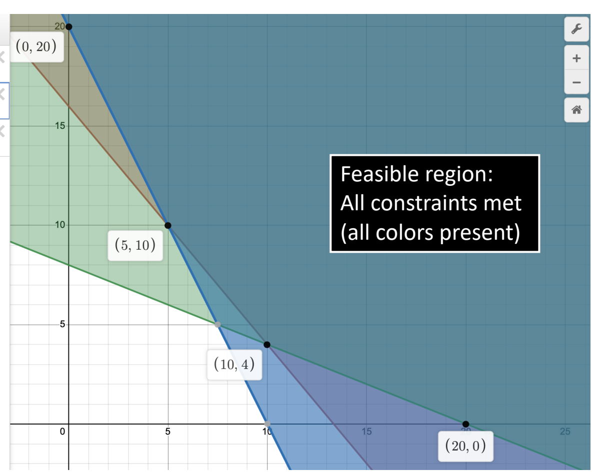


x >= 0 and y >= 0
are not shown to declutter the plot



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Example 2: infeasible LP

Additional constraint to Example 1:

- A costs \$3/oz and B costs \$4/oz
- Budget: \$40

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Why is it infeasible?

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Example 3: daily planner

Someone is making a daily planner. Outside of 10 hours of sleep every day, they want to set aside a few hours for studying and a few hours for connecting with friends.

- Gets 15 units/hr of payoff for studying up to 3 hours and 10 units/hr of payoff after 3 hours of studying (basically, brain slows down).
- Gets 20 units/hr of payoff from connecting with friends.
- Wants at least 6 hours of study/day
- Wants at most 6 hours of time with friends/day

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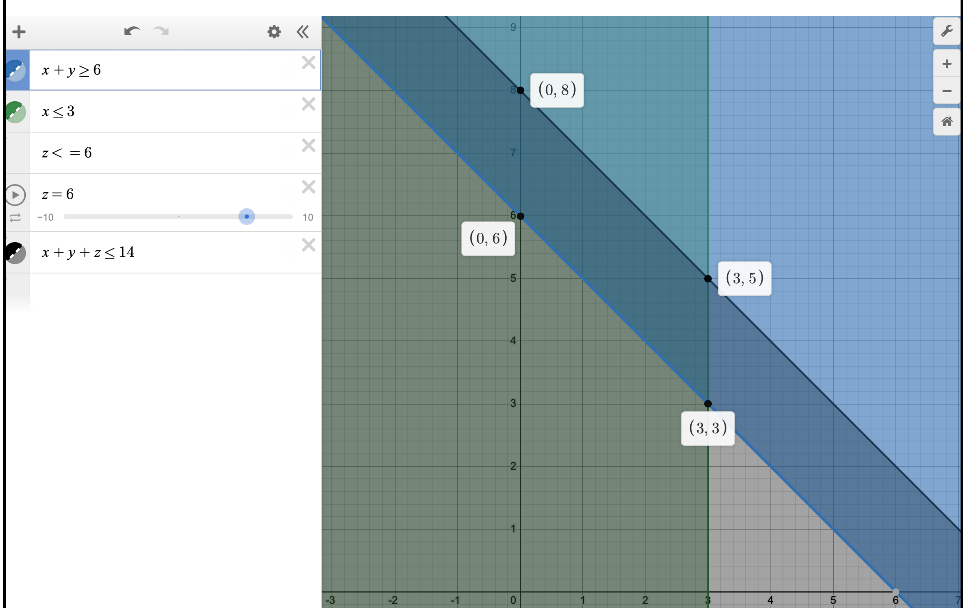

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Unbounded LP

- Objective function can be made arbitrarily good while satisfying all constraints
- Change Example 1 to make it unbounded

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Example 4: unbounded LP

- A tennis player is making a plan for practicing service and volley. She gets a payoff of 10 from every service and 5 from every volley.
- She wants to practice service at least 100 times a day and doesn't want to practice volleys more than 500 times a day. What's her optimal plan?

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Algorithms for solving LP

- Simplex (Dantzig, 1947)
  - Worst case exponential time
  - Practically fast
- Ellipsoid (Khachiyan, 1979)
  - $O(n^4 L)$ for n variables and L input bits
  - Pseudo-polynomial
- Karmarkar's algorithm (Karmarkar, 1984)
  - $O(n^{3.5} L)$ for n variables and L input bits
  - Pseudo-polynomial, but breakthrough for practical reasons
- Open problem: strongly polynomial algorithm?

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LP duality (von Neumann, 1947)

Interview with Dantzig

http://www.personal.psu.edu/ecb5/Courses/M475W/WeeklyReadings/Week%2015/An_Interview_with_George_Dantzig.pdf

von Neumann: "I don't want you to think that I am pulling all this out of my sleeve on the spur of the moment like a magician. I have just recently completed a book with Oscar Morgenstern on the theory of games. What I am doing is conjecturing that the two problems are equivalent. The theory that I am outlining for your problem is an analogue to the one we have developed for games."

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LP duality

- If the "primal" LP is maximization, its "dual" is minimization and vice versa.
- Every variable of the primal LP leads to a constraint in the dual LP and every constraint of the primal LP leads to a variable in the dual LP.
- Dual of dual is primal.

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Definition of dual LP

Source:
Applied Mathematical
Programming book

Primal

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

Dual

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

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Definition of dual LP

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Primal

Maximize $\mathbf{c}^T\mathbf{x}$
subject to:
$A\mathbf{x} <= \mathbf{b}$
$\mathbf{x} >= 0$

Dual

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

Dual

Minimize $\mathbf{b}^T\mathbf{y}$
subject to:
$A^T\mathbf{y} >= \mathbf{c}$
$\mathbf{y} >= 0$

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Example 5: LP duality

- How many Bowdoin logs and chocolate cakes should Thorne make to maximize its revenue?

Derive primal and dual LP

- Objective function: Each log has a satisfaction of 10 (or price of $10), each cake 5.
- Constraints: For both desserts, the chef needs to use an oven, a food processor, and a boiler.

	Processing time/log	Processing time/cake	Total available time
Oven	5 min	1 min	85 min
Food processor	1 min	10 min	300 min
Boiler	4 min	6 min	120 min

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Primal

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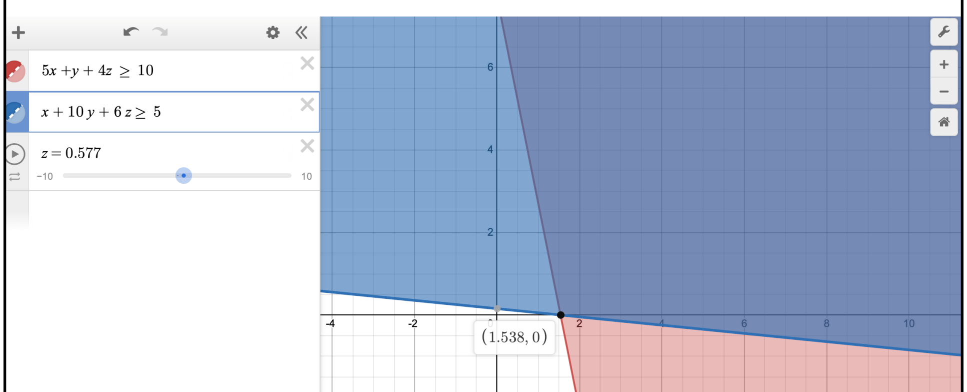
Dual



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Dual: intuition

- Moulton wants to borrow Thorne's equipment for a day for a special event.
- Moulton will pay Thorne $y1/min, $y2/min, and $y3/min for the 3 equipment, resp. such that:

  1. (Dual objective) Moulton minimizes the total cost of renting
  2. (Dual constraints) Moulton will make sure that Thorne recuperates the lost payoff for each piece of dessert through rental income

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What's the dual for Example 1: diet problem?

- A nutritionist wants to prepare a special diet for a patient. The meals should contain a minimum of 400 mg of calcium, 10 mg of iron, and 40 mg of vitamin C. The meals are to be prepared from foods A and B.
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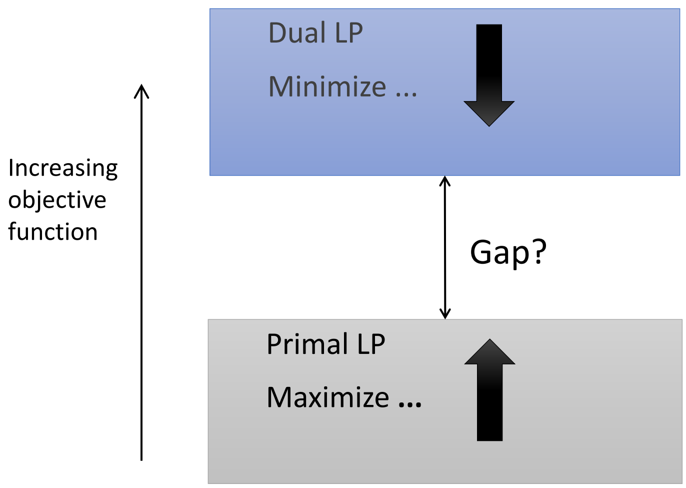
Weak duality theorem



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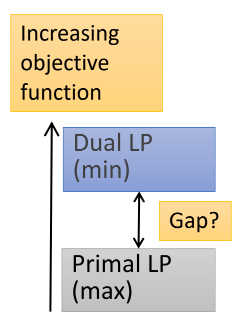
Weak duality theorem

- Any feasible solution of the dual LP (minimization) gives an upper bound on the optimal solution of the primal LP (maximization). [That's how we defined dual!]
  - Proof (next)



- Any feasible solution of the primal LP (maximization) is a lower bound on the optimal solution of the dual LP (minimization).

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Proof: weak duality theorem

Show that the primal objective <= the dual objective.

Primal

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

Dual

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

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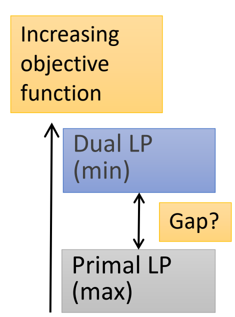
Implications: weak duality thm

- What will happen if primal (or dual) is unbounded?
- Primal unbounded ➔ Dual infeasible
- Dual unbounded ➔ Primal infeasible
- Both primal and dual may be infeasible (although not implied by this theorem)



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Strong duality theorem

If the primal LP has a finite optimal solution, then so does the dual LP. Moreover, these two optimal solutions have the same objective function value.

In other words, if either the primal or the dual LP has a finite optimal solution, the gap between them is 0.

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Complementary slackness

- If the strong duality theorem holds:
  primal constraint non-binding (not equal) =>
  corresponding dual variable = 0 at OPT
  - Similar condition holds for dual constr. & primal var.
- The reverse implication may not hold!

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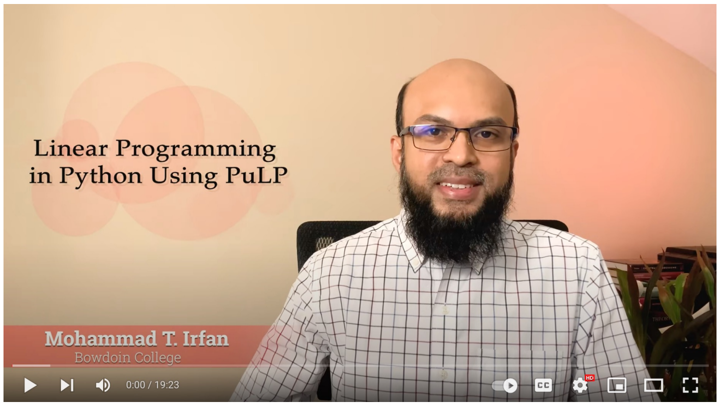
Coding LP in Python: PuLP



https://youtu.be/qa4trkLfvwQ

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2-player zero-sum game

Algorithm via LP duality

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Example 6: 2-player zero-sum game

Assumption (wlog): sum of payoffs in each cell is 0

Column player

Row player

	L	R
U	2, -2	-1, 1
D	-3, 3	4, -4

↔

	L	R
U	2	-1
D	-3	4

Matrix A

Example:
(U,L): row gains 2 and col. loses 2

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Row player

- How much gain can row player guarantee?
  - Call it v_r
  - Wants largest v_r possible
- Row: choose mixed strategy ***p*** (vector of prob.) to maximize v_r
- Expected loss of col. for playing *j*
  $= \Sigma_i (p_i A_{i,j})$

	L	R
U	2	-1
D	-3	4

Matrix A

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Row player's LP

Row player's thought process:
maximize my guaranteed gain v
knowing that column player will minimize his loss. In other words, col. player will make sure v <= col. player's loss for any of his action j.

$v_r = \max v$

subject to

$v \leq \Sigma_i (p_i A_{i,j})$, for each action j of column player

$\sum_i p_i = 1$

$p_i \geq 0$, for each action i of row player

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Column player

- How little (v_c) can col. player pay to row?
- Choose mixed strategy ***q*** (vector of probabilities) to minimize v_c
- Expected gain of <u>row</u> player for playing i $= \Sigma_j (A_{i,j}\, q_j)$

	L	**R**
U	2	-1
D	-3	4

Matrix A

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Column player's LP

Col. player's thought process:
minimize my loss (or row's gain) u
knowing that row player will choose to maximize his gain. In other words, u >= row player's gain for playing any action i.

v_c = min u
subject to

$$u \geq \Sigma_j (q_j A_{i,j}), \text{ for each action } i \text{ of row player}$$

$$\sum_j q_j = 1$$

$$q_j \geq 0, \text{ for each action } j \text{ of column player}$$

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Minimax Theorem

- At an equilibrium, $v_r = v_c$
  - Proof:
    1. The two LPs are duals of each other.
    2. Primal LP has a finite optimal solution (it's feasible + bounded).
    3. By the strong duality theorem, $v_r = v_c$.
  - Another proof:
    1. Let v* be row player's payoff at a NE.
    2. v^* >= v_r, because v_r is row player's guaranteed payoff and v^* cannot be lower than that.
    3. By assumption of NE, column player will not give row player more than v_r. So, $v_r = v^*$. SImilarly, $v_c = v^*$. Therefore, $v_r = v_c$.
- This quantity v_c or v_r is known as the value of the game (v^*)

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Another application of LP

Correlated equilibrium

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Definition: correlated equilibrium

- A probability distribution p over *action profiles* such that whenever an action profile a is drawn according to p and each player i is individually told to play a_i:
  - Playing a_i is i's best response conditioned on seeing a_i.
  - That is, for any other action a_i' of i:

$$\sum_{a_{-i}} p(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i}} p(a_i, a_{-i}) u_i(a_i', a_{-i}).$$

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Does LP work for NE?

No

Reason:

Definition 1.4.4 (Expected utility of a mixed strategy). *Given a normal-form game* (N, A, u), *the expected utility* u_i *for player* i *of the mixed-strategy profile* $s = (s_1, \ldots, s_n)$ *is defined as*

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j=1}^{n} s_j(a_j).$$

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How to compute NE?

Computational complexity

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